

Norm Linear Space Lecture Slides

For Student in University Level

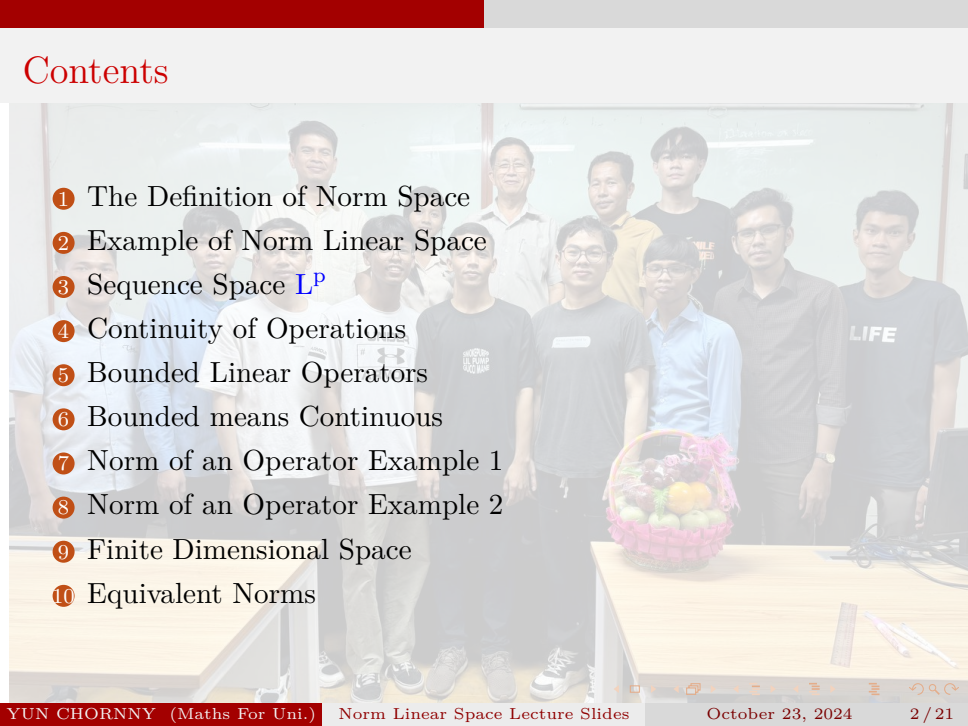
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Definition of a norm

Definition of Normed Space

Suppose X is a Vector Space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. A norm on X is a real value function $\|x\|$ with the following properties:

- ① Zero Vector: $\|x\| = 0$ if and only if $x = 0$.
 - ② Scalar factors: $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{F}$ and all $x \in X$.
 - ③ Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.
- A norm vector space $(X, \|\cdot\|)$ consists of a vector space X and a norm $\|x\|$. One generally thinks of $\|x\|$ as the length of x .
 - It is easy to check that every norms satisfies $\|x\| > 0$ for all $x \in X$.
 - Every normed vector space $(X, \|\cdot\|)$ is also a metric space (X, d) , as one may define a metric d using the formula $d(x, y) = \|x - y\|$.

Example of Normed Vector Spaces

- Given any $p \geq 1$, we can define a norm on $\mathbb{R}^k$ by letting

$$\|x\|_p = \left[\sum_{i=1}^k |x_i|^p \right]^{\frac{1}{p}}$$

- The Space $C[a, b]$ has a similar norm for any $p \geq 1$, namely

$$\|f\|_p = \left[\int_a^b |f(x)|^p dx \right]^{\frac{1}{p}}$$

- There is also a norm on $\mathbb{R}^k$ for the case $p = \infty$. It is defined by :

$$\|x\|_\infty = \max_{1 \leq i \leq k} |x_i|$$

- Finally, there is a similar norm on $C[a, b]$ which is given by :

$$\|f\|_\infty = \max_{a \leq x \leq b} |f(x)|.$$

Sequence space in L^p

- The space L^p consists of all real sequence $\mathbf{x} = \{x_n\}$ such that

$$\sum_{n=1}^{\infty} |x_n|^p < \infty.$$

- It is a normed vector space for any $p \geq 1$ and its norm is given by :

$$\|\mathbf{x}\|_p = \left[\sum_{n=1}^{\infty} |x_n|^p \right]^{\frac{1}{p}}$$

- The space L^∞ consists of all bounded real sequence $\mathbf{x} = \{x_n\}$. It is a normed vector space and its norm is given by:

$$\|\mathbf{x}\|_\infty = \sup_{n \geq 1} |x_n|$$

- The space c_0 consists of all real sequence $\{x_n\}$ which converge to 0. It is easily seen to be a subspace of L^∞ .

Continuity of Operations

Theorem 1 - Product Norm

Suppose X, Y are normed vector spaces. Then one may define a norm on the product $X \times Y$ by letting $\|(x, y)\| = \|x\| + \|y\|$

Theorem 2 - Continuity of Operations

The following functions are continuous in any normed vector space X .

- ① The norm $f(x) = \|x\|$ where $x \in X$.
 - ② The vector addition $g(x, y) = x + y$ where $x, y \in X$.
 - ③ The scalar multiplication $h(\lambda, x) = \lambda x$ where $\lambda \in \mathbb{F}$ and $x \in X$.
- We shall mainly use this theorem to justify computations such as $\lim_{n \rightarrow \infty} \|x_n\| = \left\| \lim_{n \rightarrow \infty} x_n \right\|$.

Bounded Linear Operators

Definition of Bounded - Linear and Continuous Operators

Let X, Y be normed vector spaces over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

- 1 A function $T : X \rightarrow Y$ is called a linear operator, if

$$T(x, y) = T(x) + T(y), \quad T(\alpha x) = \alpha T(x).$$

for all $x, y \in X$ and all scalars $\lambda \in \mathbb{F}$.

- 2 A function $T : X \rightarrow Y$ is called bounded, if there exists a real number $M > 0$ such that $\|T(x)\| \leq M \|x\|$. for all $x \in X$.
- 3 A function $T : X \rightarrow Y$ is called continuous, if it is continuous with respect to the metrics which are induced by the norms.

- A linear operator is also known as a linear transformation.
- By definition, every linear operator T is such that $T(0) = 0$.

Bounded means Continuous

Theorem 3. Bounded means continuous

Suppose X, Y are normed vector spaces and let $T : X \rightarrow Y$ be linear. Then T is continuous if and only if T is bounded.

Theorem 4. Norm of an operator

Suppose X, Y are normed vector spaces and let $L(X, Y)$ of all bounded, linear operators $T : X \rightarrow Y$ is itself a normed vector space. In fact, one may define a norm $L(X, Y)$ by letting

$$\|T\| = \sup_{x \neq 0} \frac{\|T(x)\|}{\|x\|}.$$

- It is easy to check that $\|T(x)\| \leq \|T\| \cdot \|x\|$ for all $x \in X$.
- One also has $\|S \circ T\| \leq \|S\| \cdot \|T\|$ whenever $S, T \in L(X, X)$.

Norm of an operator: Example 1

- Consider the right shift operator $R : L^p \rightarrow L^p$ which is defined by $R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$.
- This Operator is easily seen to be linear and we also have $\|R(x)\|_p = \|x\|_p$ for all $x \in L^p$.

In particular, the norm of this operator is equal to $\|R\| = 1$.

- The left shift operator $L : L^p \rightarrow L^p$ is similarly defined by $L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$.

Since $\|L(x)\|_p \leq \|x\|_p$ for all $x \in L^p$, we find $\|L\| = 1$. On the other hand, We also have $\|L(x)\|_p \leq \|x\|_p$ whenever $x_1 = 0$ and this implies that $\|L\| \geq 1$. We may thus conclude that $\|L\| = 1$.

Norm of an operator: Example 2

- Suppose that $T : (\mathbb{R}^n, \|\cdot\|_1) \rightarrow (\mathbb{R}^m, \|\cdot\|_\infty)$ is left multiplication by the $m \times n$ matrix A . We then have

$$\|T(x)\|_\infty = \max_i \left| \sum_j a_{ij} x_j \right| \leq \max_{i,j} |a_{ij}| \cdot \sum_j |x_j| = \max_{i,j} |a_{ij}| \cdot \|x\|_1$$

and this implies that $\|T\| \leq \max_{i,j} |a_{ij}|$.

- On the other hand, the standard unit vector $x = e_j$ satisfies

$$\frac{\|T(x)\|_\infty}{\|x\|_1} = \max_i \left| \sum_j a_{ij} x_j \right| = \max_i |a_{ij}|,$$

so we also have $\|T\| \geq \max_i |a_{ij}|$ for each j . We conclude that $\|T\| = \max_{i,j} |a_{ij}|$.

Finite - dimensional vector spaces

- Suppose that X is a vector space with basis $x_1, x_2, \dots, x_k$. Then every element $x \in X$ can be expressed as a linear combination

$$x = c_1x_1 + c_2x_2 + \dots + c_kx_k.$$

for some uniquely determined coefficients $c_1, c_2, \dots, c_k \in \mathbb{F}$.

Theorem 5 - Euclidean norm

Suppose that X is a vector space with basis $x_1, x_2, \dots, x_k$. Then one may define a norm on X using the formula

$$x = \sum_{i=1}^k c_i x_i \rightarrow \|x\|_2 = \sqrt{\sum_{i=1}^k |c_i|^2}.$$

This norm is also known as the Euclidean or standard norm on X .

Equivalent norms

Definition - Equivalent norms

We say that two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ of a norm vector space X are equivalent, if there exist constant $C_1, C_2 > 0$ such that

$$C_1 \|x\|_a \leq \|x\|_b \leq C_2 \|x\|_a \text{ for all } x \in X.$$

Theorem 6 - Equivalent of all norms

The norms of a finite - dimensional vector space X are all equivalent.

Some Exercises related with Norm Linear Space

Exercise 1

We denote $C^1([0, 1]) = \{f : [0, 1] \rightarrow \mathbb{R} : f \in C([0, 1])\}$

For $f \in C^1([0, 1])$, we defined :

$$N_1(f) = f(0) + 2 \int_0^1 |f'(x)| dx$$

$$N_2(f) = 2f(0) + \int_0^1 |f(x)| dx$$

- Prove that N_1 and N_2 are 2 norms on $C([0, 1])$.
- Do N_1 and N_2 are equivalent norms on $C([0, 1])$?

Solutions to exercise above

a). Proving that N_1 and N_2 are norms on $C([0,1])$

To show that N_1 and N_2 are norms on $C([0,1])$, we need to verify that they satisfy the three properties of norms:

1. Positivity : $N(f) \geq 0$ and $N(f) = 0$ if and only if $f = 0$.
2. Scalar multiplication: $N(\alpha f) = |\alpha|N(f)$ for any scalar α .
3. Triangle inequality: $N(f + g) \leq N(f) + N(g)$.

For $N_1(f)$:

$$N_1(f) = f(0) + 2 \int_0^1 |f'(x)| dx$$

Solutions to exercise above

1. Positivity: - Clearly, $f(0) \geq 0$ and $\int_0^1 |f'(x)| dx \geq 0$, so $N_1(f) \geq 0$.

- If $N_1(f) = 0$, then $f(0) = 0$ and $\int_0^1 |f'(x)| dx = 0$, which implies $f'(x) = 0$ almost everywhere.

Therefore, $f(x)$ must be a constant function. Since $f(0) = 0$, $f(x) = 0$ for all $x \in [0, 1]$.

Thus, $N_1(f) = 0$ if and only if $f = 0$.

Solutions to Exercise above

b). Scalar multiplication:

$$N_1(\alpha f) = (\alpha f)(0) + 2 \int_0^1 |(\alpha f)'(x)| dx = \alpha f(0) + 2|\alpha| \int_0^1 |f'(x)| dx$$

$$\implies N_1(\alpha f) = |\alpha| N_1(f).$$

3. Triangle inequality:

$$N_1(f + g) = (f + g)(0) + 2 \int_0^1 |(f + g)'(x)| dx$$

Using the triangle inequality for integrals,

$$N_1(f + g) \leq f(0) + g(0) + 2 \int_0^1 |f'(x)| dx + 2 \int_0^1 |g'(x)| dx \leq N_1(f) + N_1(g)$$

Thus, we get $N_1(f)$ is a norm on $C([0, 1])$.

Solutions to Exercise above

For $N_2(f)$:

$$N_2(f) = 2f(0) + \int_0^1 |f(x)|dx$$

1. Positivity:

- Clearly, $f(0) \geq 0$ and $\int_0^1 |f(x)|dx \geq 0$, so $N_2(f) \geq 0$.

- If $N_2(f) = 0$, then $2f(0) = 0$ and $\int_0^1 |f(x)|dx = 0$, which implies $f(0) = 0$ and $f(x) = 0$ almost everywhere in $[0, 1]$.

Since $f(x)$ is continuous, f must be zero everywhere on $[0, 1]$.

Thus, $N_2(f) = 0$ if and only if $f = 0$.

Solutions to Exercise above

2. Scalar multiplication:

$$\begin{aligned} N_2(\alpha f) &= 2(\alpha f)(0) + \int_0^1 |\alpha f(x)| dx = 2\alpha f(0) + |\alpha| \int_0^1 |f(x)| dx \\ &= |\alpha| \left(2f(0) + \int_0^1 |f(x)| dx \right) = |\alpha| N_2(f) \end{aligned}$$

3. Triangle inequality:

$$N_2(f + g) = 2(f + g)(0) + \int_0^1 |f(x) + g(x)| dx$$

Using the triangle inequality for absolute value inside the integral,

$$N_2(f + g) \leq 2(f(0) + g(0)) + \int_0^1 |f(x)| dx + \int_0^1 |g(x)| dx$$

Continue

$$N_2(f + g) \leq 2f(0) + 2g(0) + \int_0^1 |f(x)|dx + \int_0^1 |g(x)|dx = N_2(f) + N_2(g)$$

Since N_2 satisfy positivity, scalar multiplication, and the triangle inequality,

Thus N_2 is a norm on $C([0, 1])$.

b). Equivalence of N_1 and N_2

Two norms N_1 and N_2 on a vector space are called equivalent if there exist constants c_1 and c_2 such that for all $f \in C([0, 1])$,

$$c_1 N_2(f) \leq N_1(f) \leq c_2 N_2(f).$$

We will check if such constants c_1 and c_2 exist.

Comparing $N_1(f)$ and $N_2(f)$:

$$N_1(f) = f(0) + 2 \int_0^1 |f'(x)| dx$$

$$N_2(f) = 2f(0) + \int_0^1 |f(x)| dx$$

1. Upper Bound for N_1 in terms of N_2 : By the definition of N_1 and N_2 , note that:

$$N_1(f) = f(0) + 2 \int_0^1 |f'(x)| dx.$$

Since $\int_0^1 |f'(x)| dx$ is bounded by $\|f\|_1$, we can write:

$$N_1(f) \leq f(0) + 2 \int_0^1 |f(x)| dx = f(0) + 2\|f\|_1.$$

Therefore, $N_1(f) \leq 2N_2(f)$, with $c_2 = 2$.

2. Lower Bound for N_1 in terms of N_2 : For the lower bound :

$$\int_0^1 |f'(x)|dx \leq N_2(f) = 2f(0) + \int_0^1 |f(x)|dx.$$

So,

$$N_1(f) \geq f(0) + 2 \int_0^1 |f(x)|dx, \quad \text{with } c_1 = 1$$

Hence, there exist constants $c_1 = 1$ and $c_2 = 2$ such that

$$c_1 N_2(f) \leq N_1(f) \leq c_2 N_2(f). \quad (\text{rigorously proof, contact to me})$$

Thus, N_1 and N_2 are equivalent norms on $C([0, 1])$.

Therefore,

$$N_1 \text{ and } N_2 \text{ are equivalent norms on } C([0, 1]).$$