Martingale in Probability Theory

Lesson Summarizing with Exercises Correction

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Introduction to Martingale

In probability theory, a martingale is a mathematical model of a fair game. It is a sequence of random variables $(X_n)_{n \geq 0}$ that represents a fair stochastic process. Specifically, a martingale satisfies two main properties :

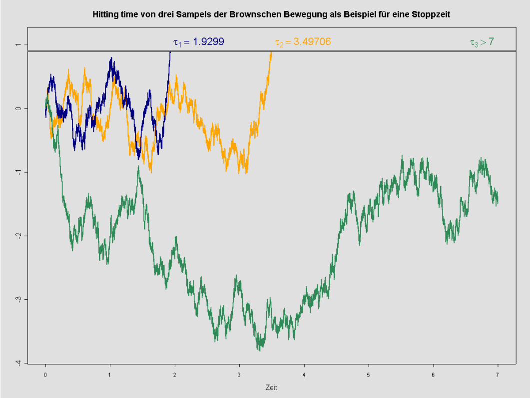
1. Integrability : Each random variable X_n in the sequence is integrable, meaning $\mathbb{E}[|X_n|] < \infty$.
2. Fair Game Condition (Conditional Expectation) : For every $n \geq 0$, the expected value of the next observation, given all past observations, is equal to the most recent observation:

$$\mathbb{E}[X_{n+1} \mid X_0, X_1, \ldots, X_n] = X_n.$$

This condition reflects that future values are expected to be the same as the present value, given the known past, indicating no predictable trends or drifts.

Introduction to Martingale

Martingales are used in various fields such as finance, economics, and gambling theory. They form a fundamental concept in probability theory and stochastic processes, playing a key role in the development of modern probability and statistical theories.



Martingale apply in Gambling

Definition of Martingale (Discrete Stochastic Process)

A discrete time process $\{X_n\}$ is called a Martingale with respect to filtration $\{\mathcal{F}_n\}$ if and only if

- $\{X_n\}$ is $\{\mathcal{F}_n\}$ - adapted (Measurable) ;
- $\mathbb{E}[|X_n|] < \infty$ i.e., Finite Expected Value : (Integrability)
- $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ almost surely.

■ Filtration $\{\mathcal{F}_n\}$ is a collection of random variables that we can control them properly.

■ In Finance Language : Filtration is a collection of information or data that we can clarify them properly.

■ Mathematical Meaning in Probability : Filtration $\{F_n\}$ is an increasing sequence of random variable.

Properties of Conditional Expectation

- $\mathbb{E}[\mathbb{E}[X|\varphi]] = \mathbb{E}[X]$
- (Measurability) : If X is $\varphi-$ measurable , $\mathbb{E}[X|\varphi] = X$
- (Linearity) : $\mathbb{E}[a_1X_1 + a_2X_2|\varphi] = a_1\mathbb{E}[X_1|\varphi] + a_2\mathbb{E}[X_2|\varphi]$
- (Positivity) : If $X \geq 0, \mathbb{E}[X|\varphi] \geq 0$
- (Tower property) : If $\mathcal{H} \subseteq \varphi, \mathbb{E}[\mathbb{E}[X|\varphi]|\mathcal{H}] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]|\varphi] = \mathbb{E}[X|\mathcal{H}]$
- (Take Out What is known) : If Z is $\varphi-$ measurable , then $\mathbb{E}[ZX|\varphi] = Z\mathbb{E}[X|\varphi]$
- (Independence) : If X is independent of φ , then $\mathbb{E}[X|\varphi] = \mathbb{E}[X]$

Remark : Martingale Property

We know above that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$

Remark : $\mathbb{E}[\mathbb{E}[X_{n+1}|\mathcal{F}_n]] = \mathbb{E}[X_n]$

Induction : $\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n] = ... = \mathbb{E}[X_0]$ " Constant "

(We called Martingale Model Fair Game)

- $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n$, So X_n is a sub martingale

  " Favorable game to you "
- $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n$, So X_n is a super martingale

  " Un favorable game to you "

Some Examples in Martingale

Example 1

Let $X_1, X_2, \ldots, X_n$ be iid $\mathcal{N}(0,1)$. We define

$\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and

$\alpha \in \mathbb{R}$ and $M_n = \exp\left\{\alpha \sum_{i=1}^{n} X_i - \frac{n\alpha^2}{2}\right\}$

Prove that $(M_n)_{n \geq 1}$ is a martingale with respect to filtration $(\mathcal{F}_n)$.

Solution to Example 1

Prove that $(M_n)_{n \geq 1}$ is a martingale with respect to $(\mathcal{F}_n)$

- M_n is $\mathcal{F}_n$ - measurable ?

Given $M_n = \exp\left\{\alpha \sum_{i=1}^{n} X_i - \frac{n\alpha^2}{2}\right\}$,

We can say that M_n is

$\mathcal{F}_n$ - measurable since X_i is $\mathcal{F}_n$ - measurable and

exponential of sum is continuous $\Longrightarrow$ M_n is $\mathcal{F}_n$ - measurable.

First condition has already completed !

Solution to Example 1

- M_n is Integrable ?

$$\begin{aligned} E\left[|M_n|\right] = E\left[M_n\right] &= E\left[e^{\alpha\sum\limits_{i=1}^{n} X_i - \frac{n\alpha^2}{2}}\right] \\ &= e^{-\frac{n\alpha^2}{2}} E\left[e^{\alpha\sum\limits_{i=1}^{n} X_i}\right] \\ &= e^{-\frac{n\alpha^2}{2}} \prod_{i=1}^{n} E\left[e^{\alpha X_i}\right] \end{aligned}$$

Since $X \sim N(0,1) \Rightarrow \text{mgf} \sim N(0,1) = e^{\frac{\alpha^2}{2}}$

$\Rightarrow E\left[M_n\right] = e^{-\frac{n\alpha^2}{2}} \times \left(e^{\frac{\alpha^2}{2}}\right)^n = 1 < \infty$

Hence M_n is Integrable .

Solution to Example 1

- (Conditional Expectation property) : $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = M_n$

We get

$$
\begin{aligned}
E\left[M_{n+1}|F_n\right] &= E\left[e^{\alpha \sum\limits_{i=1}^{n+1} X_i - \frac{(n+1)\alpha^2}{2}} |F_n\right] \\
&= E\left[e^{\alpha \sum\limits_{i=1}^{n} X_i - \frac{n\alpha^2}{2} + \alpha X_{n+1} - \frac{\alpha^2}{2}} |F_n\right] \\
&= E\left[M_n \cdot e^{\alpha X_{n+1} - \frac{\alpha^2}{2}} |F_n\right] \\
&= M_n \cdot E\left[e^{\alpha X_{n+1} - \frac{\alpha^2}{2}} |F_n\right] \\
&= M_n e^{-\frac{\alpha^2}{2}} E\left[e^{\alpha X_{n+1}} |F_n\right]
\end{aligned}
$$

Solution to Example 1

But $E\left[e^{\alpha X_{n+1}}\right] = e^{\frac{\alpha^2}{2}} \equiv \text{mgf } N \sim (0,1)$

Then we get $M_n e^{-\frac{\alpha^2}{2}} E\left[e^{\alpha X_{n+1}}\right] = M_n e^{-\frac{\alpha^2}{2}} \cdot e^{\frac{\alpha^2}{2}} = M_n$

Hence , $E\left[M_{n+1}|F_n\right] = M_n \Rightarrow M_n$ is a martingale with respect to $(F_n)_{n\geq 1}$.

Thus , $(M_n)_{n\geq 1}$ is a Martingale with respect to $\mathcal{F}_n$.

Homework

Given $X_1, X_2, \dots$ be iid and $\mathbb{P}(X_i = 2) = \frac{1}{2}$ and $\mathbb{P}(X_i = -1) = \frac{1}{2}$

Define $S_n = \sum_{i=1}^{n} X_i$, Is (S_n) a martingale with respect to natural filtration $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$?

Thank you for your attention !!