Functional Analysis

Linear Operator Lecture Slides

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contents

Definition of Operator and Linear Operator

Definition of Operator and Linear Operator

Let X and Y be two normed spaces and $T: D(T) \to Y$, where $D(T) \subset X$ be an operator ssuch that for all $x, y \in X$ and scalars α we have :

1. $T(x+y) = T(x) + T(y)$
2. $T(\alpha x) = \alpha T(x)$

Example

Given Function $T: C([0,1]) \to \mathbb{R}$ defined by $T(f) = \int_0^1 f(t)\,dt$

Prove that T is Linear Operator ?

Proof

Given I and J satisfies definition of linear operator that
$I = \int_0^1 f(t)\,dt$ and $J = \int_0^1 g(t)\,dt$
by definition of Linear Operator we get:

1. $I + J = \int_0^1 f(t)\,dt + \int_0^1 g(t)\,dt = \int_0^1 [f(t) + g(t)]\,dt$
2. suppose
$$\forall \alpha \in \mathbb{R}: \ f(\alpha t) = \int_0^1 f(\alpha t)\,dt = \alpha \int_0^1 f(t)\,dt = \alpha f(t)$$

Hence T is a Linear Operator.

Bounded Linear Operator

A bounded linear operator is defined as :

Definition

Let X and Y be two normed spaces and $T : D(T) \to Y$, where $D(T) \subset X$ be an operator ssuch that for all $x, y \in X$ and scalars α we have :

1. $T(x+y) = T(x) + T(y)$
2. $T(\alpha x) = \alpha T(x)$
3. $\exists c : \| Tx\| \leq c\|x\|$ then $c > 0$

For $x = 0$ we have $Tx = 0$ as T is linear operator. Let $x \neq 0$, $\|Tx\| \leq c\|x\|$ for all $c \in \mathbb{R}$ can be written as

$$\frac{\|Tx\|}{\|x\|} \leq c$$

and this shows that the set:

$$\left\{ \frac{\|Tx\|}{\|x\|} : x \in D(T) - \{0\} \right\}$$

is bounded above by c, in $\mathbb{R}$ therefore its supremum exists. This supremum is denoted by $\|T\|$, thus we have

$$\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \in D(T) - \{0\} \right\}$$

$$\frac{\|Tx\|}{\|x\|} \leq \|T\|, \ (x \neq 0)$$

$$\Rightarrow \|Tx\| \leq \|T\| \, \|x\|$$

By definition of a supremum $\|T\|$ is the smallest possible value of c

Remark

A bounded linear operator maps bounded set in $D(T)$ onto bounded sets in Y. This motivates the term bounded operator

Proof.

Suppose $T : X \to Y$ is bounded, there exists a real number $c > 0$ such that $\|Tx\| \leq c\|x\|$ for all $x \in X$.
Take any bounded subset A of X, there exist $M > 0$ such that $\|x\| \leq M$ for all $x \in A$,

$$\|Tx\| \leq c\|x\| \leq cM = k$$

This shows that T maps is bounded sets in X into bounded sets in Y. $\square$

before we consider general properties of bounded linear operators, let us take a look at some typical examples,

Example

Identity Operator The identity operator $I : X \to X$ on a normed space $X \neq 0$ is bounded and has norm $\|I\| = 1$. Since

$$\|Ix\| = \|x\| \leq c\|x\| \text{ for any } c \geq 1$$

$$\text{and } \|I\| = \sup\left\{\frac{\|Ix\|}{\|x\|} : x \in D(T) - \{0\}\right\}$$

$$= \sup\left\{\frac{\|x\|}{\|x\|} : x \in D(T) - \{0\}\right\}$$

$$= \sup\{1, 1, 1, ...\} = 1$$

Example

Zero Operator The Zero Operator $0 : X \to Y$ on a normed space X is bounded and has a norm $\|0\| = 0$. Since :

$$\|0x\| = \|0\| \leq c\|x\| \ , \ (c > 0) \text{ and}$$

$$\|0\| = \sup\left\{\frac{\|0x\|}{\|x\|} = 0 : x \in D(T) - \{0\}\right\}$$

$$= \sup\left\{\frac{\|0\|}{\|x\|} = 0 : x \in D(T) - \{0\}\right\}$$

$$= \sup\{0, 0, ...\} = 0$$

Example

Differentiation Operator Let X be the normed space of all polynomials on $[0,1]$ with norm $\|x\| = \max_{t\in[0,1]} |x(t)|$ a differentiation operator T is defined on X by

$$Tx = y \text{ where } y(t) = x'(t).$$

This operator is linear but not bounded.

Proof.

Suppose that differential operator T is bounded, Let $x_n \in X$ such that $x_n(t) = t^n$ where $n \in \mathbb{N}$ Then,

$$Tx_n = x_n{}' \text{ where } x_n{}'(t) = nt^{n-1}$$

$$\text{and } \|x_n\| = \max_{t\in[0,1]} |x(t)| = \max_{t\in[0,1]} |t^n| = 1$$

$$\text{and } \|Tx_n\| = \max_{t\in[0,1]} \left|x_n{}'(t)\right| = \max_{t\in[0,1]} \left|nt^{n-1}\right| = n$$

$$\text{and } \|T\| > \frac{\|Tx_n\|}{\|x_n\|} = n \in \{1, 2, 3, ...\}$$

which is a contradiction, Since $n \in \mathbb{N}$ is arbitrary.
we conclude that T is not bounded.

$\square$

Example

Integral Operator We can define an integral operator $T : C[0,1] \to C[0,1]$ by

$$Tx = y \text{ where } y(t) = \int_0^t x(T)$$

This Operator is Linear, We prove that T is bounded.

Proof.
Given $|x(T)| \leq \max_{t\in[0,1]} |x(t)| = \|x\|$ (1)

hence

$$\|Tx\| = \|y\|$$

$$= \max_{t\in[0,1]} |y(t)|$$

$$= \max_{t\in[0,1]} \left| \int_0^t x(T)\, dT \right|$$

$$= \max_{t\in[0,1]} \int_0^1 |x(T)|\, dT$$

$$\leq \max_{t\in[0,1]} \|x\| = \|x\| \text{ (using 1)}$$

The result is $\|Tx\| \leq \|x\| = c\|x\|$ that $c \geq 1$. □

Example

Matrix Operator A real matrix $A = \alpha_{ij}$ with m rows and n columns defines an operator $T : \mathbb{R}^n \to \mathbb{R}^m$ by:

$$Tx = Ax = y \quad (3)$$

where $X = (x_{ij})$, $j = 1, 2, 3, ..., n$ and $Y = (y_{ij})$, $i = 1, 2, 3, ..., m$ are column vectors. In terms of components. (3) become

$$y_i = \sum_{j=1}^{n} \alpha_{ij} x_{ij} \quad (4)$$

$i = 1, 2, 3, ..., m$, T is linear, because matrix multiplication is a linear operation. T is bounded.

Proof.

To Prove this, we first remember that the norm on $\mathbb{R}^n$ is given by

$$\|y\| = \left(\sum_{i=1}^{n} {y_i}^2\right)^{\frac{1}{2}}$$

Similarly, for $x \in \mathbb{R}^n$. Using the Cauchy-Schwarz inequality:

$$\|Tx\|^2 = \|y\|^2 = \sum_{i=1}^{m} {y_i}^2$$

$$= \sum_{i=1}^{m}\left[\sum_{j=1}^{n} \alpha_{ij} x_{ij}\right]^2 \quad \text{(using 4)}$$

$$\leq \sum_{i=1}^{m}\left[\left(\sum_{j=1}^{n} {\alpha_{ij}}^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} {x_{ij}}^2\right)^{\frac{1}{2}}\right]^2$$

$\square$

Proof.

Go On

$$\|Tx\|^2 = \sum_{i=1}^{m}\left[\left(\sum_{j=1}^{n}{\alpha_{ij}}^2\right)^{\frac{1}{2}}\|x\|\right]^2$$
$$= \sum_{i=1}^{m}\left[\left(\sum_{j=1}^{n}{\alpha_{ij}}^2\right)^{\frac{1}{2}}\right]^2\|x\|^2$$
$$= \sum_{i=1}^{m}\left(\sum_{j=1}^{n}{\alpha_{ij}}^2\right)\|x\|^2$$
$$= \|x\|^2\sum_{i=1}^{m}\sum_{j=1}^{n}{\alpha_{ij}}^2$$

Let $c^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}{\alpha_{ij}}^2$, We get

$\|Tx\|^2 \leq c^2\|x\|^2 \Rightarrow \|Tx\| \leq c\|x\|.$ □