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EQUICONVERGENCE THEOREM FOR FUNCTIONALLY DIFFERENTIAL OF THE FIRST ORDER OPERATOR ON THE GRAPH

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Abstract. This paper investigates the functional-differential operators with an involution defined on the four edges of the graph, three of which form cycle. Get boundary value problem for the resolvent operator , investigated the regularity of the boundary conditions, the necessary evaluation of the resolvent. Received teoerema uniform convergence of expansions in eigenfunctions of the operator with trigonometric Fourier series.

Keywords: functional-differential operators, graph, cycle, resolution, boundary, equiconvergence Fourier series.

We consider a connected geometric graph Γ, consisting of four edges, three of which form a cycle. A vector approach is used [1], when each edge of the graph is parametrized by a segment $[0,1]$, and a function on a graph is understood as a vector function $y(x) = (y_1(x), y_2(x), y_3(x), y_4(x))^T$ (T- transposition sign), component of which $y_k(x)$, corresponding to the k-th edge is a scalar function on the interval $[0,1]$. In accordance with this approach, we set on Γ the following statement:

$$(Ly)(x) = \begin{pmatrix} \alpha_1 y_1'(x) + \beta_1 y_1'(1-x) + p_{11}(x)y_1(x) + p_{12}(x)y_1(1-x) \\ \alpha_2 y_2'(x) + \beta_2 y_2'(1-x) + p_{21}(x)y_2(x) + p_{22}(x)y_2(1-x) \\ \alpha_3 y_3'(x) + \beta_3 y_3'(1-x) + p_{31}(x)y_3(x) + p_{32}(x)y_3(1-x) \\ y_4'(x) + p(x)y_4(x) \end{pmatrix} \tag{1}$$

$$(Ly)(x) = (l_1(y_1), l_2(y_2), l_3(y_3), l_4(y_4))^T$$

$$y_4(0) = y_2(1), y_3(0) = y_4(1), y_2(0) = y_3(1), y_1(0) = y_4(0), \tag{2}$$

where $\alpha_k^2 < \beta_k^2, k = \overline{1,3}, p_{kj}(x) \in C^1[0,1]$ .Boundary conditions (2)are generated by the continuity condition at internal nodes Γ . Thus, we will consider the case of a graph of four edges, when functional differential operators are given on three edges, including the edge that is not included in the cycle, and on one edge, an ordinary differential operator of the first order. Operator (1) with general boundary conditions $U(y) = 0$ belongs to the class of functional-differential operators with involution, the study of which has received intensive development.

The paper investigates the issues of convergence of expansions in terms of eigenfunctions and associated functions of the operator . An equiconvergence theorem with a trigonometric Fourier series is obtained.

The method of counter integration of the resolvent of the operator is used L . Namely, if $S_r(f,x)$ - partial sum of the Fourier series of the function $f(x)$ by eigenfunctions and associated functions of the operator, including terms for which $|\lambda_k| \leq r$, λ_k -eigenvalues, then

$$S_r(f,x) = -\frac{1}{2\pi i}\int_{|\lambda|=r}(R_\lambda f)(x)d\lambda$$

Where R_λ - operator resolvent L

1. To construct and study a boundary value problem for R_λ Let's use some results from [8]. Let $y = R_\lambda f$ where $R_\lambda = (L - \lambda E)^{-1}$ — operator resolvent L, $y(x) = (y_1(x), y_2(x), y_3(x), y_4(x))^T$, $f(x) = (f_1(x), f_2(x), f_3(x), f_4(x))^T$. Then for $y(x)$ we have the following system

$$\alpha_1 y_1'(x) + \beta_1 y_1'(1-x) + p_{11}(x)y_1(x) + p_{12}(x)y_1(1-x) = \lambda y_1(x) + f_1(x), \tag{3}$$

$$\alpha_2 y_2'(x) + \beta_2 y_2'(1-x) + p_{21}(x)y_2(x) + p_{22}(x)y_2(1-x) = \lambda y_2(x) + f_2(x), \tag{4}$$

$$\alpha_3 y_3'(x) + \beta_3 y_3'(1-x) + p_{31}(x)y_3(x) + p_{32}(x)y_3(1-x) = \lambda y_3(x) + f_3(x), \tag{5}$$

$$y_4'(x) + p(x)y_4(x) = \lambda y_4(x) + f_4(x), \tag{6}$$

subject to boundary conditions (2).

Consider the following boundary value problem in the space of vector functions of dimension 7:

$$Qz'(x) + P(x)z(x) = \lambda z(x) + m(x), \tag{7}$$

$$M_0 z(0) + M_1 z(1) = 0, \tag{8}$$

where $Q = diag(Q_1, Q_2, Q_3, Q_4),\ P(x) = diag(P_1(x), P_2(x), P_3(x), P_4(x))$,

$$Q_k = \begin{pmatrix} \alpha_k & -\beta_k \\ \beta_k & -\alpha_k \end{pmatrix}, Q_1 = \begin{pmatrix} \alpha_1 & -\beta_1 \\ \beta_1 & -\alpha_1 \end{pmatrix}, Q_2 = \begin{pmatrix} \alpha_2 & -\beta_2 \\ \beta_2 & -\alpha_2 \end{pmatrix}, Q_3 = \begin{pmatrix} \alpha_3 & -\beta_3 \\ \beta_3 & -\alpha_3 \end{pmatrix}, Q_4 = 1$$

$$P(x) = diag(P_1(x), P_2(x), P_3(x), P_4(x)),\ \text{где}\ P_k(x) = \begin{pmatrix} P_{k1}(x) & P_{k2}(x) \\ P_{k2}(1-x) & P_{k1}(1-x) \end{pmatrix}, k = \overline{1,3}$$

$P_4 = (p(x)), m(x) = (m_1(x), .., m_7(x))^T, m_1(x) = f_1(x), m_2(x) = f_1(1-x), m_3(x) = f_2(x), m_4(x) = f_2(1-x), m_5(x) = f_3(x), m_6(x) = f_3(1-x), m_7(x) = f_4(x), M_0, M_1$ square matrices of dimension 7, and $(M_0)_{17} = (M_0)_{25} = (M_0)_{33} = -(M_0)_{46} = (M_0)_{51} = -(M_0)_{57} = (M_0)_{66} = -(M_0)_{74} = 1, -(M_1)_{13} = -(M_1)_{27} = -(M_1)_{35} = (M_1)_{67} = (M_1)_{72} = (M_1)_{44} = 1$, other elements $(M_k)_{ij} = 0$.

Lemma 1. *If λ is such that R_λ exists and $y = R_\lambda f$, то $z(x) = (z_1(x), ..., z_7(x))^T$, where $z_1(x) = y_1(x), z_2(x) = y_1(1-x), z_3(x) = y_2(x), z_4(x) = y_2(1-x), z_5(x) = y_3(x), z_6(x) = y_3(1-x), z_7(x) = y_4(x)$, is the solution (7)-(8). Conversely, if $z(x)$ satisfies (7)-(8), the corresponding homogeneous problem has only a trivial solution, then R_λ exists and $(R_\lambda f)(x) = (y_1(x), ..., y_4(x))^T$, that $y_1(x) = z_1(x), y_2(x) = z_3(x), y_3(x) = z_5(x), y_4(x) = z_7(x)$.*

Diagonalizing the problem (7)-(8), we get

$$u'(x) + \widetilde{P}(x)u(x) = \lambda D u(x) + \widetilde{m}(x), \tag{9}$$

$$\widetilde{M_0}u(0) + \widetilde{M_1}u(1) = 0 \tag{10}$$

where $\widetilde{P}(x) = diag(\widetilde{P}_1(x), \widetilde{P}_2(x), \widetilde{P}_3(x), \widetilde{P}_4(x), \widetilde{P}_k(x) = B_k^{-1}Q_k^{-1}P_k(x)B_k$,

$$D = diag(D_1, D_2, D_3, D_4), D_k = diag(i/\sqrt{d_k}, -i/\sqrt{d_k}), d_k = \beta_k^2 - \alpha_k^2, k = \overline{1,3}, D_4 = 1,$$

$$\widetilde{m}(x) = diag(B_1^{-1}Q_1^{-1}, B_2^{-1}Q_2^{-1}, B_3^{-1}Q_3^{-1}, B_4^{-1}Q_4^{-1})m(x),$$

$\widetilde{M_k} = M_k B,\ B = diag(B_1, B_2, B_3, B_4), B_k = \begin{pmatrix} 1 & b_k \\ b_k & 1 \end{pmatrix}$, $b_k = \frac{[i\sqrt{d_k}+\alpha_k]}{\beta_k}, k = \overline{1,3}, B_4 = 1.$

To obtain asymptotic estimates for the solution of the boundary value problem (9)-(10) the system is being transformed (9), replacing a nonzero matrix $\widetilde{P}(x)$ to a matrix with elements $O(\lambda^{-1})$ ([9], with.48-58).

Let's put $H_0(x) = diag(H_{01}(x), H_{02}(x), H_{03}(x), H_{04}(x)), H_{01}(x) = diag(h_1(x), h_2(x))$ $H_{02}(x) = diag(h_3(x), h_4(x)), H_{03}(x) = diag(h_5(x), h_6(x)), H_{04}(x) = (h_7(x)), h_i(x) = exp\{-\int_0^x \widetilde{p}_{ii}(t)dt\}_{i=\overline{1,7}}$ and $\widetilde{p}_{ii}$ —diagonal matrix elements $\widetilde{P}(x)$; $H_1(x) = diag(H_{11}(x), H_{12}(x), H_{13}(x), H_{14}(x))$, where $H_{14} \equiv 0$ a $H_{1k}(x)(k = \overline{1,3})-$ codigonal matrix , which is the only solution to the matrix equation: $H'_{0k}(x) + \widetilde{P}_k(x)H_{0k} + (H_{1k}(x)D_k - D_kH_{1k}(x)) = 0$. Since the elements of the matrix $P(x)$ and correspondingly $\widetilde{P}(x)$ belong to space $C^1[0,1]$, then the elements $H_1(x)$ out of space $C^1[0,1]$, a $H_0(x)$ - from $C^2[0,1]$.

Theorem 1. *transformation $z(x,\lambda) = BH(x,\lambda)v(x)$, where $H(x,\lambda) = H_0(x) + \lambda^{-1}H_1(x)$, leads the system (7)-(8) to the mind*

$$v'(x) + P(x,\lambda)v(x) = \lambda Dv(x) + m(x,\lambda), \tag{11}$$

$$M_{0\lambda}v(0) + M_{1\lambda}v(1) = 0 \tag{12}$$

where $P(x,\lambda) = \lambda^{-1}H^{-1}(x,\lambda)[H'_1(x) + \widetilde{P}(x)H_1(x)], m(x,\lambda) = H^{-1}(x,\lambda)\widetilde{m}(x)$,

$$M_{0\lambda} = M_0BH(0,\lambda), M_{1\lambda} = M_1BH(1,\lambda).$$

2. Consider first a boundary value problem:

$$w'(x) = \lambda\widetilde{D}w(x) + m(x,\lambda), \tag{13}$$

$$U_1 = \widetilde{M_{0\lambda}}w(0) + \widetilde{M_{1\lambda}}w(1) = 0 \tag{14}$$

$$\widetilde{M_0} = M_0BH_0(0), \widetilde{M_1} = M_1BH_0(1)$$

where $m = (m_1, ..., m_7), m_i(x) \in L[0,1], \widetilde{D} = diag(\omega_1, ..., \omega_7)$

$$\omega_1 = i/\sqrt{d_1}, \omega_2 = -i/\sqrt{d_1}, \omega_3 = i/\sqrt{d_2}, \omega_4 = -i/\sqrt{d_2}, \omega_5 = i/\sqrt{d_3}, \omega_6 = -i/\sqrt{d_3}, \omega_7 = 1.$$

General system solution (13) looks like

$$w'_k(x) = \lambda\omega_kw_k(x) + m_k(x), k = \overline{1,7} \tag{15}$$

Decision of the relevant (15) homogeneous equation is

$$w_k(x) = c_ke^{\lambda\omega_kx}.$$

The method of variation of an arbitrary constant is used to find the general solution (15)

$$w_k(x) = c_k e^{\lambda\omega_k x} + \int_0^x e^{\lambda\omega_k(x-t)} m_k(t)dt. \tag{16}$$

If $Re\lambda\omega_k \geq 0$, then the second term has an exponential growth at $|\lambda| \to \infty$.

In this case, the transformation is applied $\int_0^x = \int_0^1 - \int_x^1$. Then

$$w_k(x) = \widetilde{c}_k e^{\lambda\omega_k x} - \int_0^x e^{\lambda\omega_k(x-t)} m_k(t)dt. \tag{17}$$

To write the solution of the equation (13) in vector-matrix form, we introduce the following matrices and vectors: $V(x,\lambda) = diag(e^{\lambda\omega_1 x}, ..., e^{\lambda\omega_1 x}), c = (c_1, ..., c_7)^T$ — arbitrary Vector, $g(x,t,\lambda) = diag(g_1(x,t,\lambda), ..., g_7(x,t,\lambda)), g_k(x,t,\lambda) = \varepsilon(x,t)e^{\lambda\omega_k(x-t)}$, if $Re\lambda\omega_k \leq 0, g_k(x,t,\lambda) = -\varepsilon(x,t)e^{\lambda\omega_k(x-t)}$, if $Re\lambda\omega_k \geq 0, \varepsilon(x,t) = 1$, if $x \geq t, \varepsilon(x,t) = 0$ if $x \leq t$. Then the general solution of the system (13) looks like:

$$w(x,\lambda) = V(x,\lambda) + \int_0^x g(x,t,\lambda)m(t)dt. \tag{18}$$

Subordinating it to the boundary conditions (14) we obtain the following result.

Lemma 2. *If such that the matrix* $\Delta(\lambda) = U(V(x,\lambda)) = \widetilde{M_0}V(0,\lambda) + \widetilde{M_1}V(1,\lambda)$ *is reversible, then the boundary value problem (13)-(14) is uniquely solvable for any* $m(x)$ *with components from* $L[0,1]$, *and its solution is:*

$$w(x,\lambda) = R_{1\lambda}m(x) = -V(x,\lambda)\Delta^{-1}(\lambda)U(g_\lambda m(x)) + g_\lambda m(x), \tag{19}$$

Where $g_\lambda m(x) = \int_0^1 g(x,t,\lambda)m(t)dt, U(g_\lambda m(x)) = \int_0^1 U_x(g(x,t,\lambda))m(t)dt$, *(*$U_x$ *means that* U *It applies to* g *by variable* x*).* From the boundedness of the matrix components $g(x,t,\lambda)$ then should

Lemma 3. *There are estimates:*

$$\|g_\lambda m(x)\|_\infty = O(\|m\|_1), \|U(g_\lambda m(x))\|_\infty = O(\|m\|_1),$$

Where $\|.\|_\infty$ $(\|.\|_1)$ *— norm in the space of a vector function of dimension 7 with components from* $L_\infty(L_1)$.

Just like in [8] it can be shown that for the determinant $\delta(\lambda) = det\Delta(\lambda)$ the following asymptotic representation is valid:

$$det\Delta(\lambda) = A_1(\lambda)e^{\lambda(\omega_1+\omega_3+\omega_5+\omega_7)} + A_2(\lambda)e^{\lambda(\omega_1+\omega_3+\omega_5)} + A_3(\lambda)e^{\lambda(\omega_2+\omega_4+\omega_6)}+$$

$$+A_4(\lambda)e^{\lambda(\omega_2+\omega_4+\omega_6+\omega_7)} + \sum_{k=5}^{45} A_k(\lambda)e^{\lambda(a_k\omega_1+b_k\omega_3+c_k\omega_5+d_k\omega_7)}$$

Where $A_k(\lambda) = \nu_k+O(\lambda^{-1}), k = \overline{1,4}, \nu_1 = b_1b_2b_3h_3(1)h_5(1)h_7(1), \nu_2 = -(b_1b_2b_3)^2h_3(1)h_5(1)h_7(1)$ $\nu_3 = -h_2(1)h_4(1)h_6(1), \nu_4 = -b_1b_2b_3h_2(1)h_4(1)h_6(1)h_7(1)$, and $\nu_k \neq 0; A_k(\lambda) = O(1)$, $k = \overline{5,45}, a_k\omega_1 + b_k\omega_3 + c_k\omega_5 + d_k\omega_7$ - various combinations, (numbers a_k, b_k, c_k, d_k is 0, 1, or -1 .)

Next, we assume $Re\lambda \geq 0, Im\lambda \geq 0$ that , (other cases are treated similarly). Then $det\Delta(\lambda) = e^{\lambda(\omega_2+\omega_4+\omega_6+\omega_7)}T(\lambda)$, where

$$T(\lambda) = A_4(\lambda) + A_1(\lambda)e^{-2\lambda(\omega_2+\omega_4+\omega_6)} + A_3(\lambda)e^{-\lambda\omega_7} + \sum_{k=5}^{44} A_k(\lambda)e^{\lambda(a_k\omega_1+b_k\omega_3+c_k\omega_5+d_k\omega_7)}$$

is a quasipolynomial that has a countable number of zeros ([10, c. 113, lemma 1]) , all of them are in stripes along the imaginary and real axes, and in any rectangles $|Re\lambda - t| \leq 1, |Im\lambda - t| \leq 1$ corresponding bands, their number is limited by some constant independent of t. Let us cut out these zeros from the complex plane together with circular neighborhoods of the same radius δ_0. Denote the resulting region S_{δ_0}. Then in S_{δ_0} at $Re\lambda \geq 0, Re\lambda\omega \geq 0$ fair assessment $|\delta(\lambda)| \geq c\left|e^{\lambda(\omega_2+\omega_4+\omega_6+\omega_7)}\right|$.

Hence and from the evaluation of the elements of the matrices $\Delta^{-1}(\lambda)$ and $V(x,\lambda)$ then

Lemma 4. *If $x \in [\delta, 1-\delta]$ $(\delta \in [0, 1/2))$, then the components $\eta_{ij}((x,\lambda))_{i,j=1}^7$ matrices $V(x,\lambda)\Delta^{-1}(\lambda)$, in area S_{δ_0} have the following scores: $\eta_{ij}(x,\lambda) = O(e^{-\lambda\omega_k}), i = \overline{1,7}$, $j = \overline{1,7}$.*

With the help of Lemmas 2, 3 and 4, as well as Theorem 2 from [6], proves

Lemma 5. *In area S_{δ_0} at large $|\lambda|$ estimates are the right hand side fair*

$$\|R_{1\lambda}m\|_\infty = O(\|m\|_1), \tag{20}$$

$$\|R_{1\lambda}\varphi\|_\infty = O(\lambda^{-1}), \tag{21}$$

Where $\varphi(x)$ vector function, each component of which is a function of bounded variation.

Lemma 6. *In the area of S_{δ_0} at the large $|\lambda|$ boundary value problem (11)-(12) is uniquely solvable, and for its solution $w(x,\lambda) = R_{1\lambda}m$ fair assessment*

$$\|R_{1\lambda}m\|_\infty = O(\|m\|_1), \|R_{1\lambda}m\varphi(x)\|_\infty = O(1/\lambda),$$

where $\varphi(x)$ - vector-function, each component, which is a function of limited variation.

Proof. According to (19)

$$w(x,\lambda) = R_{1\lambda}m(x) = -V(x,\lambda)\Delta_1^{-1}(\lambda)U_1(g_\lambda m(x)) + g_\lambda m(x).$$

By virtue of Lemma 4, the components of the matrix $V(x,\lambda)\Delta^{-1}(\lambda)$ for $Re\lambda w_k \geq 0$ There is $O(1)$.

From here and from the lemma3

$$\|R_{1\lambda}m\|_\infty = O(\|U(g_\lambda m(x))\|_\infty) + O(\|g_\lambda m(x)\|_\infty) = O(\|m\|_1),$$

Let now $m(x) = \varphi(x)$ — function , each component of which has limited variation. Components $g_\lambda\varphi(x)$ can be written in the form: $\int_x^1 e^{\lambda(x-t)\omega_k}\varphi(t)dt = \int_0^x e^{-\lambda(x-t)\omega_k}\varphi(t)dt$, $Re\lambda\omega_k \geq 0$ $\int_0^x e^{\lambda(x-t)\omega_k}\varphi(t)dt = \int_x^1 e^{-\lambda(x-t)\omega_k}\varphi(t)dt$, $Re\lambda\omega_k \leq 0, k = \overline{1,7}$ $\varphi(x)$ we now consider to be a scalar function of bounded variation. We transform the first of these integrals using the integration-by-parts formula,

$$\int_x^1 e^{-\lambda(x-t)\omega_k}\varphi(t)dt = -\frac{1}{\lambda\omega_k}\int_x^1 d(e^{-\lambda(x-t)\omega_k})\varphi(t) =$$
$$= -\frac{1}{\lambda\omega_k}\left[\varphi(1)e^{\lambda\omega_k(x-t)} - \varphi(x) + \int_x^1 e^{\lambda\omega_k(x-t)}d\varphi(t)\right].$$

From here, using the boundedness of the exponent and the total variation $V_0^1\varphi = \int_0^1 |d\varphi(t)|$ we have an estimate

$$\int_x^1 e^{\lambda\omega_k x}\varphi(t)dt = O(1/\lambda) + O(1/\lambda\int_0^1 |d\varphi(t)|) = O(1/\lambda).$$

Similar estimates hold for the rest of the integrals. Thus,

$$g_\lambda\varphi(x) = O(\frac{1}{\lambda}), U(g_\lambda\varphi(x)) = O(\frac{1}{\lambda}).$$

The lemma is proven.

Lemma 7. *If the components* $\widetilde{m}(x)$ *are functions of bounded variation, then*

$$\|R_{1\lambda}m - R_{2\lambda}m\|_{\infty} = O(\lambda^{-1}\|m\|_1), \|R_{1\lambda}\varphi - R_{2\lambda}\varphi\|_{\infty} = O(\lambda^{-2}).$$

From Lemma 7, by the Banach-Steinhaus theorem, it follows

Lemma 8. *For any function vector* $m(x) = (m_1(x), ..., m_7(x))^T$ *there is a ratio*

$$\lim_{r\to\infty}\left\|\int_{|\lambda|=r}\left[H(x,\lambda)\nu(x,\lambda) - H_0(x)R_{1\lambda}(H_0^{-1}m)\right]d\lambda\right\|_{\infty} = 0,$$

(integration is carried out over contours that lie entirely in S_{δ_0}*).*

3. To obtain the main result of the article (equiconvergence theorem), consider the following boundary value problem:

$$u'(x) = \lambda Du(x) + m(x), U_0(u) = u(0) - u(1) = 0. \tag{22}$$

Let us remove from, together with the circular neighborhoods of the radius, the eigenvalues ??of the scalar boundary value problems: $u_k'(x) = \lambda w_k u_k(x) + m_k(x), u_k(0) - u_k(1) = 0, k = \overline{1,7}$, and the resulting area will again be denoted S_{δ_0} . Then in S_{δ_0} problem (22) is uniquely solvable and its solution is $R_{0\lambda}m$, which has the same form (19) as $R_{1\lambda}m$, with replacement U on U_0.

Lemma 9. *If* $\widetilde{m}(x)$ *the same as in Lemma 8, then*

$$\lim_{r\to\infty}\left\|\int_{|\lambda|=r}H(x,\lambda)\left[R_{1\lambda}H_0^{-1}\widetilde{m} - R_{0\lambda}H_0^{-1}\widetilde{m}\right]d\lambda\right\|_{C[\varepsilon,1-\varepsilon]} = 0,$$

Where $\varepsilon \in (0, 1/2)$, *and* $\|.\|_{C[\varepsilon,1-\varepsilon]}$ *- norm in the space of continuous vector functions on* $[\varepsilon, 1-\varepsilon]$. The proof repeats the proof of Lemma 13 from [6].

Lemma 10. *For any function* $f(x) = (f_1(x), f_2(x), f_3(x), f_4(x))^T$ *with components from* $L[0,1]$

$$\lim_{r\to\infty}\left\|\int_{|\lambda|=r}\left[H(x,\lambda)\nu(x,\lambda) - H_0(x)R_{0\lambda}\left(H_0^{-1}\widetilde{m}\right)\right]d\lambda\right\|_{C[\varepsilon,1-\varepsilon]} = 0, \tag{23}$$

where is the solution of the problem (11)-(12), and the same as in (9).

Proof. Let us denote the integral in (16) through I. Let's represent it as a sum $I = I_1 + I_2 + I_3$, where

$$I_1 = \int_{|\lambda|=r}H(x,\lambda)\left[\nu(x,\lambda) - R_{1\lambda}H_0^{-1}\widetilde{m}\right]d\lambda, I_2 = \int_{|\lambda|=r}H(x,\lambda)\left[R_{1\lambda}H_0^{-1}\widetilde{m} - R_{0\lambda}H_0^{-1}\widetilde{m}\right]d\lambda,$$

$I_3 = \int_{|\lambda|=r}(H(x,\lambda) - H_0(x))R_{0\lambda}H_0^{-1}\widetilde{m}d\lambda$. By Lemmas 8 and 9, $\|I_1\|_\infty \to 0, \|I_2\|_{C[\varepsilon,1-\varepsilon]} \to 0$ at $r \to \infty$ .So that $H(x,\lambda) - H_0(x) = O(\lambda^{-1})$ and $R_{0\lambda}H_0^{-1}\widetilde{m} = O(\|f\|_1)$., then $\|I_3\|_\infty = O(\|f\|_1)$. If $f_i(x), (i = 1,2,3,4)$ are continuous functions of faceted variation, that $R_{0\lambda}H_0^{-1}\widetilde{m} = O(\lambda^{-1})$ and $\|I_3\|_\infty = O(r^{-1})$. Then, by the Banach-Steinhaus theorem $\|I_3\|_\infty \to 0$, at $r \to \infty$. This implies the assertion of the lemma. □

Denote by $S_r(f,x)$ partial sum of the Fourier series of the function f according to s.p.f. operator L for eigenvalues λ_k, falling into a circle $\|\lambda_k\| < r$; through $\delta_{r_j}(f_j,x)$ partial sum of the Fourier series of the function f_j according to the trigonometric system $\left\{e^{2k\pi ix}\right\}_{k=-\infty}^{+\infty}$, including terms for which $|2\pi k| < r_j$.

Theorem 2. *For any vector function $f(x) = (f_1(x), f_2(x), f_3(x), f_4(x))^T$ with components $f_i(x)$ from $L[0,1]$, and for all $\varepsilon \in (0,1/2)$*

$$\lim_{r\to\infty} \left\|S_r(f,x) - (\delta_{r_1}(f_1,x), \delta_{r_2}(f_2,x), \delta_{r_3}(f_3,x), \delta_{r_4}(f_4,x),)^T\right\|_{C[\varepsilon,1-\varepsilon]} = 0,$$

Where $r_k = r/\sqrt{d_k}, r_4 = r, k = \overline{1,3.}, d_k = \beta_k^2 - \alpha_k^2$

Proof . From the lemma 1 $R_\lambda f = (z_1(x,\lambda), z_3(x,\lambda), z_5(x,\lambda), z_7(x,\lambda))^T$,

where z_i - problem solution components (7)-(8). Taking into account Theorem 1 and Lemma 10,

$$(S_r(f,x))_1 = -\frac{1}{2\pi i}\int_{|\lambda|=r} z_1(x,\lambda)d\lambda = -\frac{1}{2\pi i}\int_{|\lambda|=r} I_\lambda(x)d\lambda + o(1) \qquad (24)$$

where I_λ is the first component of the vector $BH_0(x)R_{0\lambda}H_0^{-1}(x)\widetilde{m}$ a $o(1) \to 0$ at $r \to \infty$ evenly across $x \in [\varepsilon, 1-\varepsilon]$. Compute I_λ . We have

$$I_\lambda(x) = (BH_0(x)R_{0\lambda}H_0^{-1}(x)B^{-1}Q^{-1}m(x))_1,$$

где $m(x) = (f_1(x), f_1(1-x), f_2(x), f_2(1-x), f_3(x), f_3(1-x), f_4(x))^T$. Assuming

$$BH_0(x) = (\gamma_{ij}(x))_{i,j=1}^7, H_0^{-1}(x)B^{-1} = (\delta_{ij}(x))_{i,j=1}^7$$

and given that

$$R_{0\lambda} = diag(R^0_{\lambda\omega_1}, R^0_{\lambda\omega_2}, ..., R^0_{\lambda\omega_7}),$$

(R^0_λ - resolvent of a scalar operator $L_0y = y', y(0) = y(1)$) ,we find

$$I_\lambda = \gamma_{11}R^0_{\lambda\omega_1}(\delta_{11}g_1 + \delta_{12}g_2) + \gamma_{12}R^0_{\lambda\omega_2}(\delta_{21}g_1 + \delta_{22}g_2),$$

where $g_1(x) = d_1^{-1}[-\alpha_1 f_1(x) + \beta_1 f_1(1-x)], g_2(x) = d_1^{-1}[-\beta_1 f_1(x) + \alpha_1 f_1(1-x)]$. So that

$$-\frac{1}{2\pi i}\int_{|\lambda|=r}(R^0_{\lambda\omega_1}f)(x)d\lambda = \frac{\sqrt{d_1}}{i}\sigma_{r_1}(f,x), -\frac{1}{2\pi i}\int_{|\lambda|=r}(R^0_{\lambda\omega_2}f)(x)d\lambda = -\frac{\sqrt{d_1}}{i}\sigma_{r_1}(f,x)$$

where $r_1 = r/\sqrt{d_1}$. So we get

$$-\tfrac{1}{2\pi i}\int_{||=r} I_\lambda(x)d\lambda = \tfrac{\sqrt{d_1}}{i}\left\{\gamma_{11}[\sigma_{r_1}(\delta_{11}g_1,x) + \sigma_{r_1}(\delta_{12}g_2,x)] - \gamma_{12}[\sigma_{r_1}(\delta_{21}g_1,x) + \sigma_{r_1}(\delta_{22}g_2,x)]\right\} \tag{25}$$

. Further, using the principle of localization and the Steinhaus theorem([11], c 111), easy to get what if $a(x)$ satisfies the first-order Lipschitz condition, then

$\|\sigma_r(af,x) - a(x)\sigma_r(f,x)\|_{C[\varepsilon,1-\varepsilon]} \to 0$, at $r \to \infty$ for any function $f(x) \in L[0,1]$. Hence, (25) will go to

$$-\frac{1}{2\pi i}\int_{|\lambda|=r} I_\lambda(x)d\lambda = \frac{1}{i\sqrt{d_1}}[\theta_1\sigma_{r_1}(f_1,x) + \theta_2\sigma_{r_1}(\widetilde{f}_1,x)] + o(1) \tag{26}$$

,

Where $\theta_1 = \alpha_1(\gamma_{12}\delta_{21} - \gamma_{11}\delta_{11}) + \beta_1(\gamma_{12}\delta_{22} - \gamma_{11}\delta_{12}), \theta_2 = \alpha_1(\gamma_{11}\delta_{12} - \gamma_{12}\delta_{22}) + \beta_1(\gamma_{11}\delta_{11} - \gamma_{12}\delta_{21})$, $\widetilde{f}_1(x) = f(1-x)$. By directly calculating the elements $\gamma_{ij}(x)$ and $\delta_{ij}(x)$, find that $\theta_1 = i/\sqrt{d_1}, \theta_2 = 0$. Thus, from (24) and (26) we have $(S_r(f,x))_1 = \sigma_{r_1}(f_1,x) + o(1)$. Similarly, it is proved that $(S_r(f,x))_2 = \sigma_{r_2}(f_2,x) + o(1)$ where $r_2 = rd/\sqrt{d_1} = (r\sqrt{d_1})/\sqrt{d_1}\sqrt{d_2} = r/\sqrt{d_2}$, at uniformly across $x \in [\varepsilon, 1-\varepsilon]$. For the third, fourth, components $(S_r(f,x))$ we have $(S_r(f,x))_3 = \sigma_{r_3}(f_3,x) + o(1)$, $(S_r(f,x))_4 = \sigma_{r_4}(f_4,x) + o(1)$. This implies the assertion of the theorem.$\square$

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